

External Flows

2.1 BLASIUS SOLUTION

Blasius [1] presented a similarity solution of boundary layer equation. This was primarily based on a series expansion.

The simplest example of boundary layer flow is flow over a flat plate as shown in Fig. 2.1.

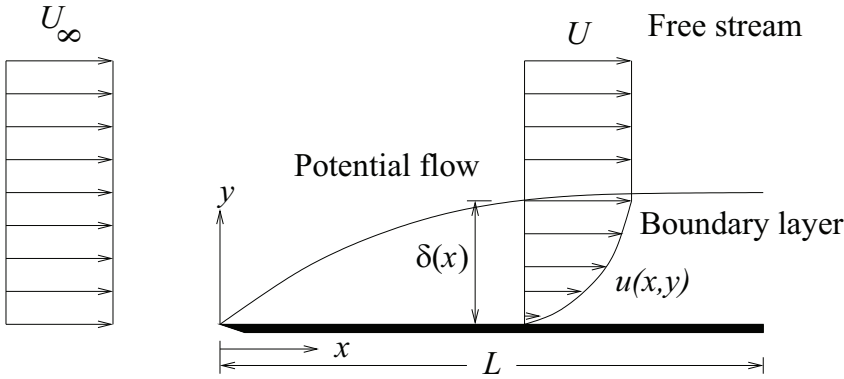


Figure 2.1 Flat plate boundary layer flow

The governing equations are:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad (2.1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2.2)$$

The boundary conditions are : (a) at $y = 0, u = v = 0$, (b) at $y = \infty, u = U_\infty$.

Substitution of $-\frac{1}{\rho} \frac{dp}{dx}$ in the boundary layer momentum equation in terms of free stream velocity produces $U_\infty (dU_\infty/dx)$ which is equal to zero. Hence the

governing equation does not contain any pressure gradient term. The characteristics parameters of this problem are u, U_∞, ν, x, y . Before we proceed further, let us discuss laws of similarity.

- The u component of velocity has the property that two velocity profiles of $u(x, y)$ at different x locations differ only by a scale factor.
- The velocity profiles $u(x, y)$ at all values of x can be made same if they are plotted in coordinates which have been made dimensionless with reference to the scale factors.
- The local free stream velocity $U(x)$ at x is an obvious scale factor for u , because the dimensionless $u(x)$ varies with y between zero and unity at all sections. The scale factor for y , denoted by $g(x)$, is proportional to local boundary layer thickness so that y itself varies between zero and unity. Finally

$$\frac{u[x_1, (y/g(x_1))]}{U(x_1)} = \frac{u[x_2, (y/g(x_2))]}{U(x_2)} \quad (2.3)$$

Again, let us consider the statement of the problem

$$u = u(U_\infty, \nu, x, y) \quad (2.4)$$

Five variables involve two dimensions. Hence it is reducible for a dimensionless relation in terms of 3 quantities. For boundary layer equations a special similarity variable is available and only two such quantities are needed. When this is possible, the flow field is said to be self similar. For self similar flows the x-component of velocity has the property that two profiles of $u(x, y)/U_\infty$ at different x locations differ only by a scale factor that is at best a function of x .

$$\frac{u}{U_\infty} = F\left(\frac{y}{\delta}\right) = F\left(\frac{y}{\sqrt{\frac{\nu x}{U_\infty}}}\right) \quad (2.5)$$

For the Blasius problem the similarity law is

$$\frac{u}{U_\infty} = F\left[\frac{y}{\sqrt{\frac{\nu x}{U_\infty}}}\right] = F(\eta) \quad (2.6)$$

where

$$\eta = \frac{y}{\delta}, \quad \delta \sim \sqrt{\frac{\nu x}{U_\infty}} \quad (2.7)$$

or

$$\eta = \frac{y}{\sqrt{\frac{\nu x}{U_\infty}}} \quad (2.8)$$

$$\psi = \int u dy = \int U_{\infty} F(\eta) \sqrt{\frac{\nu x}{U_{\infty}}} d\eta = \sqrt{U_{\infty} \nu x} \int F(\eta) d\eta \quad (2.9)$$

or

$$\psi = \sqrt{U_{\infty} \nu x} f(\eta) + C(x) \quad (2.10)$$

where $f(\eta) = \int F(\eta) d\eta$, and $C(x) = 0$ if the stream function at the solid surface is set equal to 0.

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta} \frac{\partial \eta}{\partial y} = U_{\infty} f'(\eta) \quad (2.11)$$

$$v = - \left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial \eta} \frac{\partial \eta}{\partial x} \right) = \frac{1}{2} \sqrt{\frac{\nu U_{\infty}}{x}} \left[\eta f'(\eta) - f(\eta) \right] \quad (2.12)$$

Similarly,

$$\frac{\partial u}{\partial x} = - \frac{U_{\infty}}{2} \frac{\eta}{x} f''(\eta) \quad (2.13)$$

$$\frac{\partial u}{\partial y} = U_{\infty} \sqrt{\frac{U_{\infty}}{\nu x}} f''(\eta) \quad (2.14)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_{\infty}^2}{\nu x} f'''(\eta) \quad (2.15)$$

Substituting these terms in Eq. (2.1) and simplifying we get

$$2f'''(\eta) + f(\eta)f''(\eta) = 0 \quad (2.16)$$

The boundary conditions are : at $y = 0, u = 0$ and $v = 0$. As a consequence, we can write

$$\text{at } \eta = 0, f'(\eta) = 0 \text{ and } f(\eta) = 0$$

Similarly at $y = \infty, u = U_{\infty}$ results in

$$\text{at } \eta = \infty, f'(\eta) = 1$$

Equation (2.16) is a third order nonlinear differential equation. Blasius obtained this solution in the form of a series expanded around $\eta = 0$.

Let us assume a power series expansion (for small values of η) of $f(\eta)$

$$\begin{aligned}
 f(\eta) &= A_0 + A_1\eta + \frac{A_2}{2!}\eta^2 + \frac{A_3}{3!}\eta^3 + \frac{A_4}{4!}\eta^4 + \dots \\
 f'(\eta) &= A_1 + A_2\eta + \frac{A_3}{2!}\eta^2 + \frac{A_4}{3!}\eta^3 + \dots \\
 f''(\eta) &= A_2 + A_3\eta + \frac{A_4}{2!}\eta^2 + \frac{A_5}{3!}\eta^3 + \dots \\
 f'''(\eta) &= A_3 + A_4\eta + \frac{A_5}{2!}\eta^2 + \frac{A_6}{3!}\eta^3 + \dots
 \end{aligned} \tag{2.17}$$

Boundary conditions at $\eta = 0$, $f'(\eta) = 0$ and at $\eta = 0$, $f(\eta) = 0$ applied to above will produce $A_0 = 0$; $A_1 = 0$.

We derive another boundary condition from the physics of the problem: $y = 0$, $(\partial^2 u / \partial y^2) = 0$ which leads to $\eta = 0 : f'''(\eta) = 0$; invoking this into above we get $A_3 = 0$. Finally Eq. (2.16) is substituted for f, f'', f''' into the Blasius equation, we find

$$\begin{aligned}
 &2 \left[A_4\eta + \frac{A_5}{2!}\eta^2 + \frac{A_6}{3!}\eta^3 + \dots \right] + \left[\frac{A_2}{2!}\eta^2 + \frac{A_4}{4!}\eta^4 + \frac{A_5}{5!}\eta^5 + \dots \right] \\
 &\quad \times \left[A_2 + \frac{A_4}{2!}\eta^2 + \frac{A_5}{3!}\eta^3 \right] = 0
 \end{aligned}$$

Collecting different powers of η and equating the corresponding coefficients equal to zero, we obtain

$$2A_4 = 0, \quad 2\frac{A_5}{2!} + \frac{A_2^2}{2!} = 0, \quad 2\frac{A_6}{3!} = 0$$

$$2\frac{A_7}{4!} + \frac{A_2A_4}{4!} + \frac{A_2A_4}{2!2!} = 0$$

$$2\frac{A_8}{5!} + \frac{A_2A_5}{5!} + \frac{A_2A_5}{2!3!} = 0$$

This will finally yield : $A_4 = A_6 = A_7 = 0$

$$A_5 = -\frac{A_2^2}{2}, \quad A_8 = \frac{11}{4}A_2^3, \quad A_{11} = -\frac{375}{8}A_2^4$$

Now substituting the series for $f(\eta)$ in terms of η and A_2

$$f(\eta) = \frac{A_2}{2!}\eta^2 - \frac{1}{2}\frac{A_2^2}{5!}\eta^5 + \frac{11}{4}\frac{A_2^3}{8!}\eta^8 - \frac{1}{8}\frac{375}{11!}A_2^4\eta^{11}$$

$$f(\eta) = A_2^{1/3} \left[\frac{(A_2^{1/3}\eta)^2}{2!} - \frac{1}{2}\frac{(A_2^{1/3}\eta)^5}{5!} + \frac{11}{4}\frac{(A_2^{1/3}\eta)^8}{8!} - \frac{1}{8}\frac{375(A_2^{1/3}\eta)^{11}}{11!} + \dots \right] \tag{2.18}$$

or

$$f(\eta) = A_2^{1/3} F(A_2^{1/3} \eta)$$

Equation (2.18) satisfies boundary conditions at $\eta = 0$. Applying boundary conditions at $\eta = \infty$, we have

$$\lim_{\eta \rightarrow \infty} \left[A_2^{2/3} F'(A_2^{1/3} \eta) \right] = f'(\infty) = 1$$

or

$$A_2 = \left[\frac{1}{\lim_{\eta \rightarrow \infty} F'(\eta)} \right]^{3/2} \quad (2.19)$$

The value of A_2 can be determined numerically to a good degree of accuracy. Howarth [2] found $A_2 = 0.33206$.

Numerical Approach

Let us rewrite Eq. (2.16)

$$2f'''(\eta) + f(\eta) f''(\eta) = 0$$

as three first order differential equations in the following way:

$$f' = G \quad (2.20)$$

$$G' = H \quad (2.21)$$

$$H' = -\frac{1}{2}fH \quad (2.22)$$

The condition $f(0) = 0$ [or $\eta = 0, f(\eta) = 0$] remains useful. $f'(0) = 0$ [or $\eta = 0, f'(\eta) = 0$], means $G(0) = 0$. Finally $\eta = \infty, f'(\eta) = 1$ gives $\Rightarrow G(\infty) = 1$.

Note that the equations for f and G have initial values. The value for $H(0)$ is not known. This is an unusual initial value problem. We handle this as an initial value problem by choosing values of $H(0)$ and solving by numerical methods $f(\eta), G(\eta)$ and $H(\eta)$. In general $G(\infty) = 1$ will not be satisfied for the function G arising from the numerical solution. We then choose other initial values of H so that we find an $H(0)$ which results in $G(\infty) = 1$. This method is called the shooting technique. In Eqs. (2.20) through (2.22) the primes refer to differentiation with respect to η . The integration steps following a Runge-Kutta method are given below

$$f_{n+1} = f_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \quad (2.23)$$

$$G_{n+1} = G_n + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4) \quad (2.24)$$

$$H_{n+1} = H_n + \frac{1}{6} (m_1 + 2m_2 + 2m_3 + m_4) \quad (2.25)$$

as one moves from η_n to $\eta_{n+1} = \eta_n + h$. The values of k, l and m are as follows

$$\begin{aligned}\frac{df}{d\eta} &= f' = F_1(f, G, H, \eta) \\ \frac{dG}{d\eta} &= G' = F_2(f, G, H, \eta) \\ \frac{dH}{d\eta} &= H' = F_3(f, G, H, \eta)\end{aligned}\quad (2.26)$$

$$\begin{aligned}k_1 &= h F_1(f_n, G_n, H_n, \eta_n) \\ l_1 &= h F_2(f_n, G_n, H_n, \eta_n) \\ m_1 &= h F_3(f_n, G_n, H_n, \eta_n)\end{aligned}\quad (2.27)$$

$$\begin{aligned}k_2 &= h F_1\left(f_n + \frac{1}{2}k_1, G_n + \frac{1}{2}l_1, H_n + \frac{1}{2}m_1, \eta_n + \frac{h}{2}\right) \\ l_2 &= h F_2\left(f_n + \frac{1}{2}k_1, G_n + \frac{1}{2}l_1, H_n + \frac{1}{2}m_1, \eta_n + \frac{h}{2}\right) \\ m_2 &= h F_3\left(f_n + \frac{1}{2}k_1, G_n + \frac{1}{2}l_1, H_n + \frac{1}{2}m_1, \eta_n + \frac{h}{2}\right)\end{aligned}\quad (2.28)$$

In a similar way k_3, l_3, m_3 and k_4, l_4, m_4 are calculated following standard formulae for Runge-Kutta integration.

The functions F_1, F_2 and F_3 are G, H and $-fH/2$. Then at a distance $\Delta\eta$ from the wall, we have

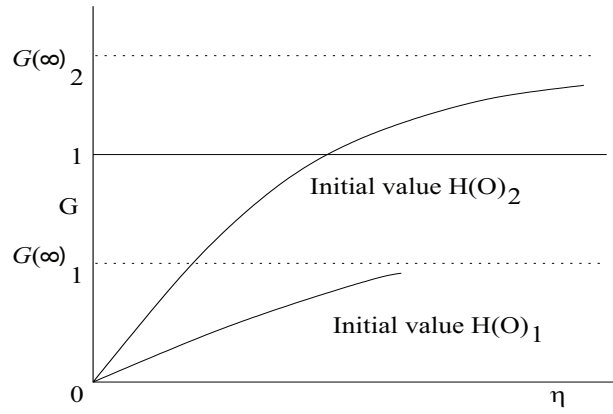
$$\begin{aligned}f(\Delta\eta) &= f(0) + G(0)\Delta\eta \\ G(\Delta\eta) &= G(0) + H(0)\Delta\eta \\ H(\Delta\eta) &= H(0) + H'(0)\Delta\eta \\ H'(\Delta\eta) &= -\frac{1}{2} f(\Delta\eta) H(\Delta\eta)\end{aligned}\quad (2.29)$$

As mentioned, $f''(0) = H(0)$ is unknown. $H(0) = S$ must be determined such that the condition $f'(\infty) = G(\infty) = 1$ is satisfied. The condition at infinity is usually approximated at a finite value of η (around $\eta = 10$).

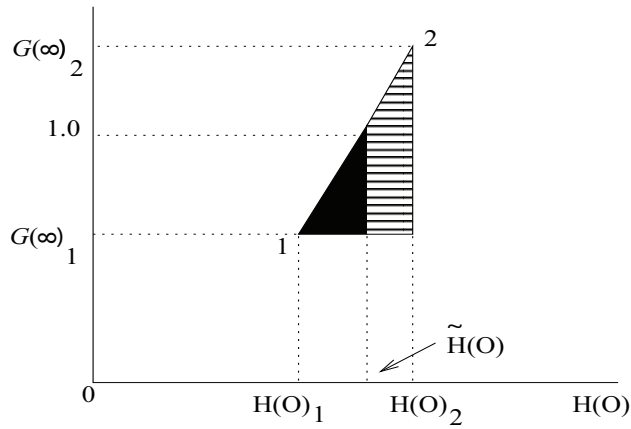
The value of $H(0)$ now can be calculated by finding $\tilde{H}(0)$ at which $G(\infty)$ crosses unity (Fig. 2.2). Refer to Fig. 2.2(b)

$$\frac{\tilde{H}(0) - H(0)_1}{1 - G(\infty)_1} = \frac{H(0)_2 - H(0)_1}{G(\infty)_2 - G(\infty)_1}$$

and repeat the process by using $\tilde{H}(0)$ and better of two initial values $H(0)$. Thus the correct initial value will be determined.



(a)



(b)

Figure 2.2 Correction of the initial guess for $H(0)$

Shear Stress

$$\tau_{wall} = \mu \frac{\partial u}{\partial y} \Big|_{y=0}$$

or

$$\tau_{wall} = \mu U_{\infty} \frac{\partial}{\partial \eta} f'(n) \frac{\partial \eta}{\partial y} \Big|_{\eta=0}$$

or

$$\tau_{wall} = \mu U_{\infty} [0.33206] \frac{1}{\sqrt{\nu x / U_{\infty}}}$$

or

$$\tau_{wall} = \frac{0.332 \rho U_{\infty}^2}{\sqrt{Re_x}}$$

Each time examine $G(\eta)$ versus η for proper $G(\infty)$. Compare the values with Schlichting [3]. The values are available in Table 2.1

Table 2.1

η	f	$f' = G$	$f'' = H$
0	0	0	0.33206
0.2	0.00664	0.06641	0.33199
0.4	0.02656	0.13277	0.33147
0.8	0.10611	0.26471	0.32739
1.2	0.23795	0.39378	0.31659
1.6	0.42032	0.51676	0.29667
2.0	0.65003	0.62977	0.26675
2.4	0.92230	0.72899	0.22809
2.8	1.23099	0.81152	0.18401
3.2	1.56911	0.87609	0.13913
3.6	1.92954	0.92333	0.09809
4.0	2.30576	0.95552	0.06424
4.4	2.69238	0.97587	0.03897
4.8	3.08534	0.98779	0.02187
5.0	3.28329	0.99155	0.01591
8.8	7.07923	1.00000	0.00000

Local skin friction coefficient

$$C_{fx} = \frac{\tau_{wall}}{\frac{1}{2}\rho U_{\infty}^2}$$

$$C_{fx} = \frac{0.664}{\sqrt{Re_x}}$$

Total friction force per unit width

$$F = \int_0^L \tau_{wall} dx = \int_0^L \frac{0.332 \rho U_{\infty}^2}{\sqrt{\frac{U_{\infty}}{\nu}}} \frac{dx}{\sqrt{x}} = \left[\frac{0.332 \rho U_{\infty}^2 x^{1/2}}{\sqrt{\frac{U_{\infty}}{\nu}} \frac{1}{2}} \right]_0^L$$

$$F = 0.664 \rho U_{\infty}^2 \sqrt{\frac{\nu L}{U_{\infty}}}$$

Average skin friction coefficient

$$\overline{C}_f = \frac{F}{\frac{1}{2}\rho U_{\infty}^2 L} = \frac{1.328}{\sqrt{Re_L}}$$